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The high-temperature susceptibility and spin–spin correlation function of the three-dimensional Ising model

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Abstract. The series analysis protocol developed in the preceding paper is applied to the susceptibility and correlation function of the three-dimensional Ising model. For the spin-$\frac{1}{2}$ Ising data, we argue that the protocol needs modification to take account of trends in the data, which trends arise from the nature of the series. In this way we obtain $\gamma = 1.239 \pm 0.003$ for all cubic lattices and for a variety of spin values. The diamond lattice data lies outside this range, and we argue that this series is effectively far shorter than the others. We also estimate $\nu = 0.632^{+0.002}_{-0.003}$ for the BCC lattice and obtain critical point estimates for all lattice and spin models studied.

1. Introduction

For more than ten years the presence of discrepancies between renormalisation group estimates of the critical exponent γ, which describes the divergence of the susceptibility of the $n = 1$ realisation of the $O(n)\phi^4$ field theory, and the series analysis estimates obtained for the $S = \frac{1}{2}$ Ising model, has been a matter of considerable concern. A partial resolution of these discrepancies has been given for certain lattices, notably the FCC (McKenzie 1983) and the BCC (Nickel 1982), but both the SC lattice and the diamond lattice series have generally yielded estimates for γ which are significantly greater than the FCC and BCC lattice values, or the field theory predictions. The situation is carefully reviewed by Gaunt (1982). Very recently, George and Rehr (1986) have given good evidence of consistent exponent estimates for the three cubic lattices, though at a somewhat lower value of γ than that found here.

What is generally noticed in the series analysis results is a variation of estimates of γ with lattice coordination number q. As q increases, so does γ apparently decrease, and it is at the high q end of the spectrum that agreement with field theory values is attained.

It has been argued that there is no reason to prefer these values to the low q series values, such as the diamond lattice ($q = 4$) estimates, particularly as the diamond lattice series is the longest—measured in numbers of series coefficients—of the available series.

In this paper we have re-examined the available series using the protocol developed in the previous paper (Guttmann 1987, hereafter referred to as I), based on the method of integral approximants. We find complete consistency in our estimates of γ for the three cubic lattices, while only the diamond lattice gives anomalously high estimates.

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We argue however that the diamond series is *effectively* the shortest of the series for any of the three dimensional lattices, and further that it is the worst behaved. We also obtain estimates of the critical temperature for all lattices, a result unobtainable by field theoretic methods.

The results referred to above were obtained for the $S=\frac{1}{2}$ Ising model. Very recently, Nickel and Rehr (1986) have published long (21 term) series for the $S=1$ and $S=2$ BCC lattice Ising model. Our analysis, applied to these two series, points to a somewhat lower value of γ than that found for the three $S=\frac{1}{2}$ lattice series. There are two possible conclusions to be drawn. One is that the exponent γ is spin dependent. The other is that the series are still too short to unambiguously reveal a precise consistent lattice and spin independent value of γ. We incline strongly towards the latter view, and accordingly take into account trends, such as the tendency for estimates of γ to decrease as the order of the approximant increases for the $S=\frac{1}{2}$ data, while for the $S=2$ data the opposite trend is evident. Under the assumption that these trends persist, we quote an estimate for γ with confidence limits wide enough to encompass all the results. In this way we obtain $\gamma=1.239\pm0.003$, in agreement with an earlier, and quite distinct, analysis (Guttmann 1986).

However, let us return to consider the effective length of a series as raised earlier. The graphs that contribute to the series expansion of the high-temperature susceptibility are the magnetic graphs, those which have precisely two vertices of odd degree. The lower the lattice coordination number, the greater is the minimum number of bonds in a specified topology. This is most readily visualised in two dimensions, where the lowest-order theta graph has five bonds on the high coordination number ($q=6$) triangular lattice, but 11 bonds on the low coordination number ($q=3$) honeycomb lattice. As it is the magnetic topologies of higher cyclomatic index that asymptotically dominate the expansion, it is clear that the expansion must be sufficiently long for these graphs to possess an embedding if sensible conclusions are to be drawn from the series analysis. One obvious measure of the 'effective length' of a series is the computer time used in its generation. That is, one hour of CPU time, utilising the same algorithm, will effectively sample the choices on a given lattice to approximately the same extent. The time taken to count graphs of n bonds on a given lattice model with critical point at v_c is proportional to v_c^{-n}. For the four common three-dimensional lattices there are N_{max} terms available, where N_{max} is 22 (diamond), 19 (SC), 21 (BCC) and 15 (FCC), while the values of the critical points are approximately 0.354 (diamond), 0.218 (SC), 0.156 (BCC) and 0.102 (FCC). The *effective* length of the series we take to be proportional to $1/v_c^{N_{max}-1}$. (The -1 in the exponent arises from the symmetry associated with the first step.) Thus we find that the effective lengths are in the ratio $1:2.7\times10^2:4.6\times10^6:2.6\times10^4$ for the diamond, SC, BCC and FCC lattices, respectively. Expressed another way, the longest series is the BCC series. The FCC series would need two further terms to be of comparable length, while the SC series would need 7 or 8 further terms and the diamond lattice a full 15 additional terms. We do not claim that this is a precise measure of the relative lengths of the four series, but it certainly shows that the diamond lattice series is by far the shortest, while the BCC is undoubtedly the longest.

On the basis of these observations, it is already clear which series can be expected to be more reliable and that is the high coordination number series. It is these that already give reasonable agreement with field theory predictions.

As we show in the next section, we obtain $\gamma=1.243\pm0.002$ for all three cubic lattices and $\gamma\approx1.239$ for the $S=1$ and 2 BCC lattices. We have also analysed the series for μ_2, the second moment of the spin-spin correlation function, which gives the correlation length exponent ν, for the $S=\frac{1}{2}$ BCC lattice, and find $\nu = 0.632^{+0.002}_{-0.003}$ for this lattice only. For other lattices the μ_2 series are far shorter and we can say no more but that they are consistent with the above result (or indeed with any other value of ν in the range 0.625–0.640).

2. Analysis of series

Our method of analysis has been fully described in I. In table 1 we give a summary of critical point ($v_c = \tanh(J/kT_c)$) and critical exponent (γ) estimates for the four lattices obtained from first-order integral approximants, and for the SC and BCC lattices, also the $K=2$ approximants. As discussed in I we have primarily used first-order integral approximants, even though the expected confluent singularity structure would suggest that second-order approximants would be more appropriate. As shown in I

Table 1. Summary of critical point and critical exponent estimates for the diamond, simple-cubic, body-centred cubic and face-centred cubic spin-$\frac{1}{2}$ Ising model susceptibility series from first-($K=1$) and second-($K=2$) order approximants.

n	v_c (error)	γ (error)	l	n	v_c (error)	γ (error)	l
Diamond lattic ($K=1$)				FCC lattice ($K=1$)			
14	0.353 654 (186)	1.2398 (117)	7	10	0.101 738 (56)	1.2470 (159)	5
15	0.353 738 (102)	1.24453 (81)	8	11	0.101 721 (20)	1.2425 (65)	8
16	0.353 855 (194)	1.2538 (152)	11	12	0.101 714 (19)	1.2400 (70)	11
17	0.353 830 (127)	1.2522 (122)	13	13	0.101 724 (14)	1.2438 (59)	11
18	0.353 880 (81)	1.2568 (83)	14	14	0.101 725 (6)	1.2442 (28)	10
19	0.353 879 (64)	1.2565 (83)	15	15	0.101 721 (4)	1.2421 (21)	11
20	0.353 864 (49)	1.2550 (46)	6				
21	0.353 787 (279)	1.2471 (310)	3				
22	0.353 825 (111)	1.2520 (108)	9				
SC lattice ($K=1$)				SC lattice ($K=2$)			
12	0.218 175 (91)	1.2500 (396)	8	14	0.218 063 (80)	1.2433 (126)	4
13	0.218 207 (91)	1.2585 (128)	11	15	0.218 130 (130)	1.2486 (199)	5
14	0.218 160 (80)	1.2523 (118)	8	16	0.218 100 (12)	1.2442 (23)	4
15	0.218 097 (96)	1.2433 (159)	17	17	0.218 115 (34)	1.2464 (67)	8
16	0.218 090 (14)	1.2422 (26)	9	18	0.218 127 (28)	1.2491 (57)	7
17	0.218 103 (20)	1.2442 (43)	9	19	0.218 130 (41)	1.2493 (76)	8
18	0.218 119 (97)	1.2440 (73)	9				
19	0.218 103 (38)	1.2443 (88)	9				
BCC lattice ($S=\frac{1}{2}$, $K=1$)				BCC lattice ($S=\frac{1}{2}$, $K=2$)			
12	0.156 128 (13)	1.2495 (21)	10	13	0.156 123 (41)	1.2483 (56)	3
13	0.156 085 (71)	1.2419 (135)	10	14	0.156 125 (57)	1.2481 (69)	5
14	0.156 094 (14)	1.2440 (39)	11	15	0.156 104 (26)	1.2458 (54)	7
15	0.156 090 (13)	1.2426 (30)	11	16	0.156 098 (6)	1.2447 (16)	8
16	0.156 095 (3)	1.2438 (8)	10	17	0.156 097 (3)	1.3441 (10)	8
17	0.156 095 (5)	1.2439 (15)	9	18	0.156 094 (7)	1.2443 (30)	8
18	0.156 095 (2)	1.2438 (15)	10	19	0.156 090 (22)	1.2431 (40)	4
19	0.156 094 (2)	1.2434 (6)	7	20	0.156 094 (10)	1.2432 (28)	6
20	0.156 092 (1)	1.2428 (5)	7	21	0.156 095 (9)	1.2434 (34)	6
21	0.156 092 (1)	1.2429 (6)	8				

for the self-avoiding walk model, the first- and second-order approximants give similar exponent and critical point estimates, but the precision of the first-order approximants is substantially higher. This is also the case here, as can be seen from table 1.

Following the procedure outlined in I, we have combined the entries in table 1 to give a single estimate of γ and v_c for each lattice. These are summarised in table 3. All errors quoted are two standard deviations from the mean.

It can be seen that the three cubic lattices all give consistent estimates of γ, which may be combined to give $\gamma = 1.243 \pm 0.002$, while the diamond lattice estimate is substantially higher at $\gamma = 1.252 \pm 0.003$. However a look at the raw data of table 1 shows that, for the diamond lattice data, the asymptotic regime has not yet been reached. For $n > 18$ there is a general downward drift of estimates of γ with increasing values of n. This behaviour is also evident in the BCC lattice data for $n < 15$, the SC lattice data for $n < 15$ and there is a hint of such behaviour in the FCC data for $n < 11$. The three cubic lattices however appear to have 'settled down' to yield stable estimates of γ and v_c (we discuss this point further subsequently), while the diamond lattice data are yet to reach asymptopia. This observation is entirely consistent with our assessment, in § 1, of the relative lengths of the four series.

In table 2 we show the corresponding results for the spin-1 and spin-2 BCC Ising model series. The expansion parameter for these series is $K = J/kT$ rather than tanh K. These data indicate a somewhat lower value of γ than does the spin-$\frac{1}{2}$ data. The spin-1 estimates are very stable in the range shown and clearly indicate a value of γ around 1.2388. The spin-2 data are almost as stable, though the $K = 2$ approximants show a very clear trend for γ to increase with increasing order of the approximant. We have attempted to extrapolate this trend by linear regression against $1/n$, which yields $\gamma = 1.2383$, in good agreement with the spin-1 values.

Table 2. Summary of critical point and critical exponent estimates for the body-centred cubic spin-1 and spin-2 Ising model susceptibility series from first-($K = 1$) and second-($K = 2$) order approximants.

	$K = 1$			$K = 2$		
n	K_c (error)	γ (error)	l	K_c (error)	γ (error)	l
BCC lattice ($S = 1$)						
15	0.224 660 (14)	1.2393 (22)	12	0.224 657 (10)	1.2387 (21)	7
16	0.224 659 (4)	1.2392 (9)	10	0.224 660 (20)	1.2391 (22)	8
17	0.224 659 (6)	1.2393 (15)	11	0.224 657 (3)	1.2389 (6)	6
18	0.224 658 (8)	1.2390 (15)	12	0.224 659 (1)	1.2392 (2)	7
19	0.224 657 (2)	1.2389 (4)	10	0.224 658 (2)	1.2392 (5)	4
20	0.224 657 (1)	1.2388 (2)	7	0.224 656 (2)	1.2386 (8)	6
21	0.224 658 (4)	1.2388 (1)	9	0.224 657 (3)	1.2388 (7)	6
BCC lattice ($S = 2$)						
14	0.292 254 (13)	1.2358 (17)	12	0.293 261 (17)	1.2366 (20)	6
15	0.293 254 (9)	1.2360 (13)	12	0.293 250 (3)	1.2354 (5)	6
16	0.293 252 (3)	1.2357 (4)	8	0.293 253 (14)	1.2357 (14)	8
17	0.293 252 (3)	1.2357 (5)	8	0.293 252 (3)	1.2357 (5)	7
18	0.293 256 (2)	1.2364 (4)	9	0.293 253 (2)	1.2359 (3)	6
19	0.293 253 (3)	1.2358 (8)	10	0.293 254 (5)	1.2361 (12)	8
20	0.293 254 (1)	1.2360 (2)	8	0.293 254 (3)	1.2361 (9)	8
21	0.293 253 (3)	1.2359 (6)	10	0.293 255 (5)	1.2363 (17)	6

Table 3. Critical point and critical exponent extimates obtained from the data in tables 1 and 2.

Lattice	Critical point (error)	Exponent (error)
Diamond	$v_c = 0.353\,834\,(33)$	$\gamma = 1.2520\,(31)$
SC	$v_c = 0.218\,097\,(12)$	$\gamma = 1.2431\,(24)$
BCC $(S=\frac{1}{2})$	$v_c = 0.156\,093\,(2)$	$\gamma = 1.2433\,(3)$
$(S=1)$	$K_c = 0.224\,657\,(1)$	$\gamma = 1.2388\,(1)$
$(S=2)$	$K_c = 0.293\,254\,(1)$	$\gamma = 1.2360\,(2)$
FCC	$v_c = 0.101\,722\,(4)$	$\gamma = 1.2427\,(17)$

It is difficult to reconcile these results with the spin-$\frac{1}{2}$ results. One possibility is that γ is spin dependent. This abhorrent suggestion should only be entertained as a last resort, and we do not believe that our evidence is strong enough to advocate this possibility. It is clear that our analysis method will only give reliable error estimates if the asymptotic regime is reached. For the spin-$\frac{1}{2}$ BCC data there is some evidence that both the v_c and γ estimates are declining as n increases. For the FCC lattice there are too few terms for clear trends to be apparent as there is quite some scatter (1.240–1.244) in the last five estimates of γ. For the SC lattice, the $K=1$ approximants abruptly drop for $n>13$, and thereafter the last five entries fluctuate in the range (1.242–1.244). The $K=2$ approximants are significantly higher, though the associated errors are large enough that the $K=1$ and $K=2$ estimates overlap. Thus we see that for the longest spin-$\frac{1}{2}$ series, there is a tendency for γ to decrease with increasing order, but there is little evidence of this behaviour in the shorter series. However, had we only 16 terms in the spin-$\frac{1}{2}$ BCC lattice series, we would have also claimed to find no decrease of γ with order. Only with the full 21 term series do we see this behaviour. It appears then that there is some evidence for the following behaviour of the spin-$\frac{1}{2}$ series. There is an initial regime in which the estimates decline as γ increases, a second regime in which the estimates of γ are steady and a third regime in which a decline is again observed. The diamond lattice data are still in the first regime, the simple cubic lattice data are in the second regime, and the FCC lattice data appear to be possibly entering the third regime, while the BCC lattice data are in the third regime. For the spin-2 data there is evidence of an *increase* in estimates of γ with increasing order, while the $S=1$ data are relatively stable. This presumably indicates the minimal effect that confluent singularities have in the $S=1$ case, in agreement with the results found by Nickel and Rehr (1986).

In the light of this picture, a 'most likely' value of γ around 1.239 must be favoured. It is difficult to quote confidence limits in the light of the foregoing, but ±0.003 would encompass all the various trends and speculations.

We turn now to an analysis of the second moment of the spin-spin correlation function, μ_2 where

$$\mu_2 = \sum_i r_i^2 \langle S_0 S_i \rangle. \tag{2.1}$$

In analogy with our study of the SAW problem, we take the *term-by-term quotient* of the μ_2 series and the susceptibility series. This is not a quantity of direct physical significance, but has the advantage that the critical point is precisely known and the exponent is $1+2\nu$. Thus the quantity is analogous to $\langle R_n^2 \rangle$ for self-avoiding walks. We have analysed it in precisely the same way as we analysed $\langle R_n^2 \rangle$ in I. In table 4 we

Table 4. Summary of critical exponent estimates for the spin-$\frac{1}{2}$ body-centred cubic correlation function. The series studied is that given by term-by-term quotients of μ_2 and χ_0, biased at $x_c = 1.0$ and at $x_c = \pm 1.0$. (*a*) Biased at $x_c = 1$, (*b*) biased at $x_c = +1$ and $x_c = -1$.

(*a*)

	$K=1$		$K=2$		$K=3$	
n	Exponent (error)	l	Exponent (error)	l	Exponent (error)	l
10	2.2765 (29)	7				
11	2.2774 (25)	9	2.2794 (50)	3		
12	2.2771 (12)	11	2.2771 (3)	4		
13	2.2773 (49)	12	2.2784 (46)	6		
14	2.2747 (55)	11	2.2765 (30)	6	2.2763 (59)	4
15	2.2751 (15)	11	2.2756 (34)	7	2.2748 (12)	4
16	2.2734 (11)	9	2.2741 (68)	7	2.2743 (11)	5
17	2.2731 (13)	12	2.2743 (16)	7	2.2727 (16)	5
18	2.2728 (21)	12	2.2712 (64)	5	2.2720 (93)	5
19	2.2734 (17)	11	2.2684 (78)	5	2.2728 (69)	4
20	2.2707 (37)	8	2.2693 (48)	6	2.2666 (80)	6
21	2.2704 (19)	8	2.2706 (31)	4	2.2622 (40)	5

(*b*)

	$K=1$		$K=2$		$K=3$	
n	Exponent (error)	l	Exponent (error)	l	Exponent (error)	l
10	2.2767 (19)	9	2.2806 (65)	3		
11	2.2771 (26)	10	2.2776 (4)	4		
12	2.2768 (19)	11	2.2775 (21)	5		
13	2.2776 (19)	12	2.2776 (22)	6	2.2756 (31)	4
14	2.2767 (27)	8	2.2757 (17)	7	2.2751 (28)	4
15	2.2741 (41)	10	2.2768 (26)	6	2.2747 (12)	5
16	2.2737 (7)	11	2.2750 (41)	7	2.2730 (22)	4
17	2.2731 (6)	11	2.2735 (35)	6	2.2732 (26)	6
18	2.2727 (17)	11	2.2703 (46)	5	2.2725 (32)	5
19	2.2729 (58)	10	2.2693 (33)	8	2.2684 (92)	5
20	2.2713 (32)	8	2.2685 (3)	5	2.2707 (68)	6
21	2.2694 (23)	8	2.2672 (38)	5	2.2645 (62)	6

summarise the integral approximants. Two sets of approximants are shown, those biased at 1.0, and those biased both at ±1.0. Since the BCC is a loose-packed lattice that allows antiferromagnetic ordering, the antiferromagnetic singularity maps to −1.

As in the analogous analysis of $\langle R_n^2 \rangle$ in I, the exponent estimates decrease as the order increases. The decline is comparable to that observed in the SC lattice $\langle R_n^2 \rangle$ data in I, and as discussed there, the higher-order approximants are likely to be the most appropriate due to the complexity of the confluences caused by taking the term-by-term quotient of μ_2 and χ_0. It is clear from table 4 that the approximants biased at ±1 give estimates of greater consistency and with lower errors than those biased only at +1. The $K = 1$ approximants in both cases point to a limit <2.270. The $K = 2$ approximants are consistent with a limit <2.268 and the $K = 3$ approximants suggest that the limit is less than 2.266. By comparing the apparent rate of convergence with that found in

I for both the triangular lattice and simple-cubic lattice $\langle R_n^2 \rangle$ series, we estimate $1+2\nu = 2.264^{+0.004}_{-0.006}$ or $\nu = 0.632^{+0.002}_{-0.003}$.

3. Discussion

This study complements several recent studies of the same problem. Adler (1983), using a generalisation of the Padé method that incorporates a transformation designed to account for confluent singularities, obtained estimates of γ in the range 1.238–1.239 for the lattices considered here. That method however is a biased method in that it depends on the right choice of the correction-to-scaling exponent.

An alternative, but also biased, analysis based on second-order integral approximants has recently been carried out by George and Rehr (1986). Their study also considered other leading exponents, as well as the correction-to-scaling exponent θ. Indeed, their method of analysis involves forcing a value of $\theta = \frac{1}{2}$ onto the second-order approximants. Other methods of implicit biasing were also employed. In this way they obtained results significantly lower than ours. For the BCC lattice they find $\gamma = 1.237 \pm 0.002$, while for the FCC lattice, only a less detailed analysis was possible by their methods, yielding $\gamma \approx 1.239^{+0.002}_{-0.009}$ and for the SC lattice they found $\gamma = 1.235 \pm 0.004$.

A related analysis by Nickel and Rehr (1986) of a family of three-dimensional models which interpolate between the spin-$\frac{1}{2}$ Ising model and the Gaussian model also gave $\gamma = 1.237 \pm 0.002$ and $\nu = 0.630 \pm 0.0015$. However, analysis of the same data by Fisher and Chen (1985) gave $\gamma = 1.2395 \pm 0.0004$ and $\nu = 0.632 \pm 0.001$. An earlier analysis of George and Rehr (1984) of this data gave $\gamma = 1.2378 \pm 0.0006$ and $\nu = 0.631\,15 \pm 0.0003$.

A multi-parameter fit method was employed by Ferer and Velgakis (1983) in their study of the BCC lattice data and they obtained $\gamma = 1.242^{+0.003}_{-0.005}$ and $\nu = 0.634^{+0.003}_{-0.004}$. A criticism of this analysis has been given by Fisher and Chen (1985). A different analysis of spin-S data by Zinn-Justin (1981) was also biased by the requirement that the exponent θ be independent of S. This gave $\gamma = 1.2385 \pm 0.0015$ and $\nu = 0.6305 \pm 0.0015$. The field theory results of Baker *et al* (1976, 1978) and Le Guillou and Zinn-Justin (1980) gave $\gamma = 1.241 \pm 0.002$ and $\nu = 0.630 \pm 0.002$. Other recent values are $\nu = 0.628 \pm 0.002$ and $\gamma = 1.240 \pm 0.002$ (Roskies 1981), $\nu = 0.630 \pm 0.003$ and $\gamma = 1.237 \pm 0.003$ (Nickel and Dixon 1982) and $\nu = 0.6305 \pm 0.0025$ and $\gamma = 1.239 \pm 0.004$ (Le Guillou and Zinn Justin 1985).

Thus our analysis has given a value of γ that agrees well with most of the recent analyses. Our analysis has the advantage over most of those cited that our method involves no bias by quantities not exactly known, except that implicit in the choice of the method of analysis. There is of course a correlation between the value of the critical temperature and the critical exponent. If we knew one, we could construct biased estimates for the other. For the SC lattice, the Monte Carlo work of Pawley *et al* (1984) gave $K_c = 0.221\,654 \pm 0.000\,006$, while our estimate is $K_c = 0.221\,657 \pm 0.000\,007$, which is in excellent agreement. The alternative Monte Carlo analysis of Barber *et al* (1985) gave $K_c = 0.221\,650 \pm 0.000\,005$, slightly lower than our value, but there is some concern that this Monte Carlo result may have been affected by a biased random number generator. The central estimate of George and Rehr (1986) was identical to ours, at $K_c = 0.221\,657 \pm 0.000\,005$, while their values for other spins and other lattices are also in good agreement with ours. However, given that we have extrapolated trends to lower the values of the spin-$\frac{1}{2}$ series exponent estimates and raise the value of the spin-2 series exponent estimates, we should make corresponding adjustments to the estimates of v_c and K_c, since these are highly correlated with γ. Numerical experimentation indicates that the effect of this correlation is such as to lower the critical point estimates of the spin-$\frac{1}{2}$ lattices by 0.002%, and to increase the spin-2 critical point estimate by 0.001%. Our 'preferred values' then become

$$v_c = 0.218\,093 \text{ SC spin-}\tfrac{1}{2}$$

$$v_c = 0.156\,090 \text{ BCC spin-}\tfrac{1}{2}$$

$$v_c = 0.220\,952 \text{ BCC spin-1}$$

$$v_c = 0.285\,130 \text{ BCC spin-2}$$

$$v_c = 0.101\,720 \text{ FCC spin-}\tfrac{1}{2}.$$

Since these are preferred values, we are hesitant to quote errors, but confidence limits of 10 in the last quoted digit seems appropriately conservative.

Our estimate of ν is in good agreement with that of Fisher and Chen (1985), though our central value is slightly higher than that of George and Rehr (1986) or Nickel and Rehr (1986). Nevertheless, our error bars overlap substantially.

The major conclusion however is that we have obtained good agreement between lattices of the exponent γ, as well as reasonable agreement with the predictions of field theory. There seems little reason to doubt the conventional view that γ is both lattice and spin independent for a given dimensionality and that the ϕ^4 continuum field theory is in the same universality class as the spin-$\frac{1}{2}$ Ising model.

It would nevertheless be reassuring to have a substantially extended diamond lattice series (or perhaps some Monte Carlo work) so that the exponent estimates from this lattice too could be seen to be consistent with those of other lattices.

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